An Uncountably Infinite Collection of Dust

Natalie Curtis

Department of Mathematics and Computer Science

The Cantor Set

The Cantor Set is a set of numbers named for Georg Cantor. The set is formed by taking all the numbers in the closed interval from zero to one (the unit interval) and, in the first iteration, removing the middle third. This leaves the two outer thirds remaining. In each subsequent iteration, the middle third is removed from all remaining intervals from the previous iteration. Starting with [0,1], the first iteration yields $[0,1/3] \cup [2/3,1]$, the next iteration yields $[0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$, and the construction continues infinitely in this manner. Figure 1 represents the construction of the Cantor Set.

Figure 1

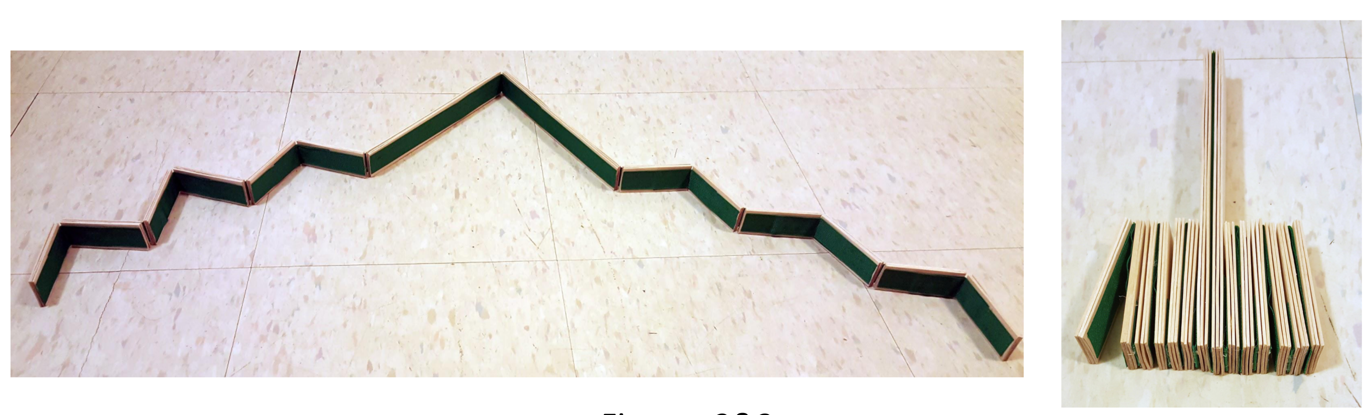
The Cantor set can be classified by four different size properties of sets: measure, density, countability, and cardinality. Due to the Cantor Set construction, these size properties oppose each other. The Cantor Set is defined as large by countability and cardinality but defined as small by measure and density.

Visual Mathematics

This project presents the size properties of the Cantor Set as three-dimensional models. By using shape, space, movement, and repetition, the models redefine the abstract, mathematical size properties and force them to exist outside of mathematical theory. The properties are demonstrated as parts of physical existence defined by spatial relationships, making them more approachable for a general audience.

Size Properties and Models

Measure (Fig. 2&3): Consider all intervals that are removed in the Cantor Set construction, all the middle thirds. At the n^{th} iteration of the construction there are 2^{n-1} open subintervals removed each having length $(1/3)^n$. The sum of all lengths of intervals removed at the n^{th} iteration is $(2^{n-1})(1/3)^n$. The sum of the lengths approaches 1 as n gets larger, so the measure of all the intervals removed from the Cantor Set is 1. The measure of the unit interval is also 1. The construction removes a total length of 1 from the starting length of 1, so the Cantor Set has measure 0. The elements in the Cantor Set are so small that they have zero length.



Figures 2&3

Density (Fig.4): The Cantor Set is nowhere dense in the real numbers. Every interval is broken apart at the subsequent iteration by the construction of the set. After infinite iterations, there are no remaining intervals, only limit points. After infinitely many iterations, there are so many spaces between elements in the Cantor Set that there is not always another element in the set between any given two. The pieces of the unit interval that remain in the Cantor Set after infinite iterations are so small they can be described as dust.

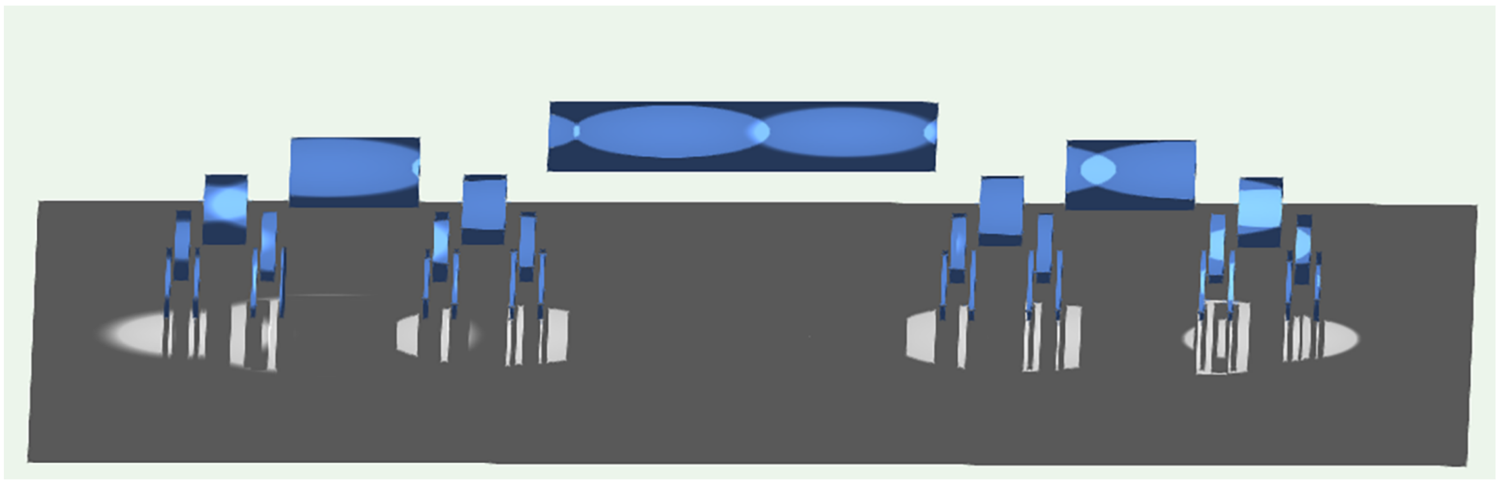


Figure 4

Size Properties and Models Cont.

Countability (Fig. 5&6): The elements in the Cantor Set are uncountable. There are more elements in the Cantor Set than all the integers or all the rational numbers. There are so many elements in the Cantor Set that it is impossible to arrange them in a way that can be counted. When attempting to arrange the elements, it is always possible to find another element in the set that was not previously listed. The number of elements in the Cantor Set is uncountably infinite.

Figures 5&6

Cardinality: The cardinality of a countably infinite set is represented as $\aleph_0$. Since uncountably infinite sets are larger than countably infinite sets, the cardinality of uncountably infinite sets is greater than $\aleph_0$. For example, the cardinality of the real numbers, which are uncountably infinite, is $2^{\aleph_0}$. The unit interval also has cardinality $2^{\aleph_0}$. Two sets have the same cardinality if a bijective function exists between them. By this property, the Cantor Set has the same cardinality as the unit interval. There are as many elements in the Cantor Set as the unit interval and the real numbers. A model of the cardinality of the Cantor Set would be created in the future work of this project.

References

Byers, William. *How Mathematicians Think.* Princeton University Press, 2007.

Vallin, Robert W. *The Elements of Cantor Sets: with Applications*. Wiley, 2013.